Sliding on the inside of a Conical Surface

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Abstract

We analyze the frictionless motion of a point-like particle that slides under gravity on an inverted conical surface. This motion is studied for arbitrary initial conditions and a general relation, valid within 13%, between the periods of radial and angular oscillations, that holds for both small and high energy trajectories, is obtained. This relation allows us to identify the closed orbits of the system. The virtues of this model to illustrate pedagogically, how in a physical system, the energy is transferred between different modes, are also emphasized. Two easy identification criteria for this type of motion with potential interest in industrial design are obtained.

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1 Introduction

The conical shape is ubiquitous in nature. Many important large-scale natural formations such as the profile of volcanoes [1, 2] or the borders of tornadoes present this universal form [3]. On a smaller scale, different devices such as silos, loudspeakers, funnels, pipes, particle precipitators or centrifugal separators have been designed by man using this geometry to perform diverse technical operations [4, 5, 6, 7]. Don't let us forget other basic devices with conical design such as wash-hand basins and lavatories. Vortices in fluids appear in some cases as evolving cone-like structures [8], and recently the avalanches of two-dimensional automata modeling the surface of a sandpile have received considerable attention from the scientific community [9]. Although the physical mechanisms behind all these complex phenomena are different, the coincidence and the ubiquity of the conical form in all of them is remarkable.

Bodies falling over inclined surfaces under the influence of gravity are also en-countered in every day life. Think, for instance, of a tobbogan, a roller coaster or a snow-board car. These systems are modeled in general physics courses as point-like particles sliding over those surfaces under the action of gravitational and surface reaction forces.

In this work, we consider the mechanical problem of a point-like particle of mass m sliding on the inside of a smooth cone of semi-vertical angle ϕ_0, whose axis points vertically upward (see Fig. 1), as a first step to the understanding of those more complicated hydrodynamical and granular systems spiraling on the inside of a conic surface. This problem is proposed by T.W.B. Kibble in his well known Classical Mechanics textbook [10]. There, the student is asked: (1) to find the Hamiltonian function, using the distance r from the vertex and the azimuth angle θ as generalized coordinates ; (2) to show that circular motion is possible for any value of r, and to

determine the corresponding angular velocity ω_0 ; (3) to deduce the angle $\bar{\phi}_0$ where the frequency of the small radial oscillations about the circular motion is also ω_0.

Setting the coordinate frame at the cone vertex, denoting the vertical direction as the OZ axis and using r and θ as the generalized coordinates, we deduce that the Hamiltonian function for this problem is

$$\mathcal{H} = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2 \sin^2 \phi_0} + mgr \cos \phi_0, \tag{1}$$

where p_r and p_θ stand for the canonically conjugated momenta of r and ϕ_0 respectively. Besides, we have assumed that the level of reference for the potential energy is $z = 0$, and the constant value of gravity is denoted by g.

As the coordinate θ does not explicitly appear in $\mathcal{H}$, θ is a *hidden* variable and from the equations of motion we deduce that p_θ is a constant of the motion. Physically this is clear because there are only two forces acting on the particle, one being gravity and the other the force of reaction exerted by the surface which is perpendicular to the surface. Thus, both forces are located in the vertical plane formed by the OZ axis and the position vector of the particle, and therefore the momentum of these forces is always horizontal and hence the vertical component of the angular momentum is conserved. In other words p_θ is a constant.

In consequence, we define the effective radial potential energy $U(r)$ for this problem as

$$U(r) = \frac{L_z^2}{2mr^2 \sin^2 \phi_0} + mgr \cos \phi_0, \tag{2}$$

where L_z is the constant value assumed for p_θ.

$U(r)$ for $L_z = 1$ and $\phi_0 = \frac{\pi}{4}$, is plotted in Fig. 2. Thus $U(r)$ is a confining potential and the particle, no matter what its energy is, cannot escape from the cone. This confinement is due to the effect of gravity for large r and to the reaction of the conical surface as a consequence of the acquired rotational kinetic energy for small r. In both cases, the conservation of p_θ limits the transference between gravitational

energy and kinetic energy as will be discussed in detail below. From Eq. (2) we deduce the radial position r_0 where U has its minimum U_0,

$$r_0 = \left[\frac{L_z^2}{m^2 g \sin^2 \phi_0 \cos \phi_0} \right]^{\frac{1}{3}}. \tag{3}$$

Thus, for any pair ϕ_0 and L_z, there exists a unique circular motion at the distance $r = r_0$, with angular velocity

$$\omega_0 = \frac{L_z}{mr_0^2 \sin^2 \phi_0}. \tag{4}$$

Now, imparting a small L_z-conserving energy perturbation Δ to this circular motion, up to an energy $E = U_0 + \Delta$, one generates an orbit with small radial oscillation around r_0 (see Fig. 2). The frequency ω_r of this oscillation is

$$w_r = \sqrt{\frac{1}{m} \left[\frac{d^2 U}{dr^2} \right]_{r=r_0}} = \sqrt{\frac{1}{m} \frac{3L_z^2}{mr_0^4 \sin^2 \phi_0}}. \tag{5}$$

Thus the radial time period $T_r = \frac{2\pi}{\omega_r}$ coincides with the period of the circular motion when $T_r = \frac{2\pi}{\omega_0}$. This condition leads to

$$\bar{\phi}_0 = \sin^{-1}(1/\sqrt{3}) \simeq 35.3^0, \tag{6}$$

which completes the answer to the three academic questions proposed at the beginning of this Section.

In Section 2, we will study the equations of motion and the trajectories of this system with arbitrary energies and will obtain a general relation between the periods of radial and angular oscillations. In Section 3, several universal geometrical and dynamical relations for a general trajectory are derived. The rich energy transfer process acting in this system is also studied. In Section 4, we state the conclusions. Finally, in the Appendix A, we will study the necessary conditions for having periodicity in a general orbit.

2   Small and Large Radial Oscillations

2.1   Equations of the system

As said in Section 1, the dynamics of the particle inside the cone can be expressed in the generalized coordinates (r, θ) (see Fig. 1). If v is the velocity of the particle, its kinetic energy $\mathcal{T}$ is written in these coordinates as

$$\mathcal{T} = \frac{1}{2} \, mv^2 = \frac{1}{2} \, m(\dot{r}^2 + r^2 \dot{\theta}^2 \sin^2 \phi_0), \tag{7}$$

where the dot means the time derivative. The potential energy $\mathcal{V}$ of the particle in the gravitational field is

$$\mathcal{V} = mgz = mgr \cos \phi_0. \tag{8}$$

Then the Lagrangian function $\mathcal{L}$ for this system in these particular coordinates appears as

$$\mathcal{L} = \mathcal{T} - \mathcal{V} = \frac{1}{2} \, m(\dot{r}^2 + r^2 \dot{\theta}^2 \sin^2 \phi_0) - mgr \cos \phi_0. \tag{9}$$

The generalized momenta are

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} \quad = \quad m\dot{r}, \tag{10}$$

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad = \quad mr^2 \dot{\theta} \sin^2 \phi_0. \tag{11}$$

If we substitute these new coordinates (p_r, p_θ) in the function

$$\mathcal{H} \;=\; \dot{r} p_r + \dot{\theta} p_\theta - \mathcal{L} \;=\; \mathcal{T} + \mathcal{V},$$

the Hamiltonian $\mathcal{H}$ shown in Eq. (1) is obtained.

The equations of motion of the particle are determined by the Lagrange's equations:

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{r}} \right) - \frac{\partial \mathcal{L}}{\partial r} \;=\; 0,$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} \;=\; 0.$$

The first of these equations gives us the evolution of the particle in the radial direction. It yields

$$\ddot{r} - r\dot{\theta}^2 \sin^2 \phi_0 + g\cos\phi_0 = 0. \tag{12}$$

And the second puts in evidence an invariant of motion because $\mathcal{L}$ is independent on the angular variable θ:

$$mr^2\dot{\theta}\sin^2\phi_0 = cte. = L_z. \tag{13}$$

This dynamical constant, L_z, is the vertical component of the angular momentum. If we substitute the value of $\dot{\theta} = L_z/(mr^2 \sin^2 \phi_0)$ in Eq. (12), the radial evolution is uncoupled from its angular dependence which only remains present through the constant L_z,

$$\ddot{r} - \left(\frac{L_z}{m\sin\phi_0}\right)^2 \frac{1}{r^3} + g\cos\phi_0 = 0. \tag{14}$$

This last equation corresponds to an integrable nonlinear oscillator in the radial direction. After integrating this motion the angular part of the dynamics is obtained through Eq. (13). Next we analyze the behavior of the radial oscillator: first for the small oscillations and then for arbitrary energy perturbations.

2.2 The radial motion

We can see at a glance the whole picture of the possible motions in the radial coordinate if the totality of its integral curves on the plane $(r, \dot{r})$ are known [11]. Equation (14) can be easily integrated since the radial and time variables are separated. If we define the parameters

$$A = \left(\frac{L_z}{m\sin\phi_0}\right)^2,$$
$$B = g\cos\phi_0,$$

and h is the constant of the first integration of Eq. (14), we obtain

$$\frac{\dot{r}^2}{2} + V(r) = h, \tag{15}$$

with

$$V(r) = \frac{A}{2r^2} + Br = \frac{U(r)}{m}, \tag{16}$$

$$h = \frac{\mathcal{H}}{m}, \tag{17}$$

where $U(r)$ is the effective radial potential energy given by Eq. (2) and $\mathcal{H}$ is the total energy of the system (Eq. (1)).

Circular motion: Depending on the initial conditions $(r, \dot{r})_{t=0}$, the energy constant h takes different values and a different equi-energy curve is drawn by the system for every h (see Fig. 3). The value $h = h_0$ for which the integral curve degenerates in an isolated point, occurs for the only minimum of the potential $V(r)$, given by the relation

$$\frac{dV(r)}{dr} = 0 \implies r = r_0 = \sqrt[3]{\frac{A}{B}},$$

where r_0 was written in Eq. (3) and represents the only existing circular orbit of the system for the actual values of L_z and ϕ_0. By substituting $L_z = mr_0^2 \omega_0 \sin^2 \phi_0$ in the expression (3), the angular frequency ω_0 of the circular motion given by Eq. (4) is found to be

$$\omega_0^2 = \frac{g \cos \phi_0}{r_0 \sin^2 \phi_0}.$$

It is curious that the angle ϕ_0 which minimizes r_0 for a L_z fixed is the complementary angle of $\bar{\phi}_0$, written in (6), that is, $\phi_0 = 54.7^0$.

General radial orbit: For $h < h_0$ motion is not possible and for $h > h_0$ the phase plane organizes itself as a pattern of closed curves nested around the equilibrium point $r = r_0$ (Fig. 3). This is a consequence of the confining property of $V(r)$, which means

$$\lim_{r \downarrow 0} V(r) \sim \frac{A}{2r^2} \xrightarrow{r \to 0} +\infty,$$

$$\lim_{r \uparrow \infty} V(r) \sim Br \xrightarrow{r \to \infty} +\infty.$$

Hence, given the initial conditions of the system, the particle inside the cone is oscillating in the radial coordinate between the minimum, r_{min}, and the maximum, r_{max}, values of its radial trajectory (Fig. 2). Taking into account that $r_0^3 = A/B$ and that $V(r)$ reaches the same value at these extreme points, $V(r_{min}) = V(r_{max})$, we obtain a global relation of a particular trajectory:

$$\frac{2\,r_{min}^2 r_{max}^2}{r_{min} + r_{max}} = r_0^3. \tag{18}$$

2.3 Small oscillations

For a value h_1 slightly bigger than h_0, the motion of the system is represented approximately on the phase plane $(r, \dot{r})$ by an ellipse around the singular point $r = r_0$. In this case, r_{min} and r_{max} are of the same order of magnitude as r_0 and the relation (18) becomes simplified to

$$r_{min} r_{max} \simeq r_0^2. \tag{19}$$

If we put $r = r_0 + \rho$, with ρ small, and we linearize the expression (15), the equation of a harmonic oscillator is derived:

$$\ddot{\rho} + \omega_r^2 \rho \simeq 0, \tag{20}$$

where

$$\omega_r^2 = \left[\frac{d^2 V}{dr}\right]_{r=r_0} = \frac{3g \cos \phi_0}{r_0},$$

as was advanced in Eq. (5). This approximation is valid, provided the energy of the perturbation $\Delta = m(h_1 - h_0)$ is mainly stored in the radial harmonic oscillation. This is verified when the condition $\frac{\Delta}{m\omega_r^2 \rho^2} \sim 1$ is fulfilled.

The relation between this radial frequency and the frequency of the circular motion is

$$\frac{w_r^2}{w_0^2} = 3 \sin^2 \phi_0. \tag{21}$$

If we define the respective time periods, $T_r = \frac{2\pi}{w_r}$ and $T_0 = \frac{2\pi}{w_0}$, the latter expression implies that

$$\frac{T_0}{T_r} = \sqrt{3} \sin \phi_0,$$

which indicates that the angular part of the motion is faster than the radial when $\phi_0 < \bar{\phi}_0$. In this range, the particle will perform many revolutions around the cone axis for each radial oscillation.

An orbit will represent a periodic motion, and thus, a closed trajectory in the four-dimensional phase space $(r, \theta, \dot{r}, \dot{\theta})$, when the ratio ω_r/ω_0 is a rational number (see Appendix A). In the case of small radial oscillations the perturbed circular orbit will be periodic if there exists a pair (p, q) of integers verifying

$$\frac{p^2}{q^2} = 3 \sin^2 \phi_0. \tag{22}$$

This means that for each pair (p, q), with $p < \sqrt{3}\, q$, there is an angle ϕ_0 of the cone opening for which the perturbed circular orbits are periodic. If this last relation (22) is not satisfied for any pair (p, q), the dynamics is quasi-periodic and fill up the torus on which it develops. As an example, if $p = q = 1$, then $\phi_0 = \bar{\phi}_0 \simeq 35.3^0$ (Eq. (6)) and the small radial-oscillating trajectories are ellipses on the cone.

The wave shape of the perturbed circular orbit in the plane (θ, r) is not sinusoidal as can be checked in Fig. 4. The trajectory covers a longer θ-angular distance when its radial coordinate is under the value $r = r_0$ than when it is in the upper region of r_0. A look at the first order approximation to the orbit explains this fact. If we write

$$r(t) \simeq r_0 + \rho \cos(\omega_r t),$$

then, from Eq. (13), we obtain

$$\theta(t) \simeq \omega_0 t - 2 \left(\frac{\omega_0}{\omega_r}\right) \left(\frac{\rho}{r_0}\right) \sin(\omega_r t).$$

A whole oscillation in r is made when θ runs on the interval $\frac{\omega_0}{\omega_r} \cdot [0, 2\pi]$. If the perturbation in the θ-coordinate is not considered, the orbit $r(\theta)$ is sinusoidal and satisfies $r(\theta) < r_0$ when $\theta \in \frac{\omega_0}{\omega_r} \cdot \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. Actually, a simple calculation shows that the small perturbation forces the dynamics to stay under $r = r_0$ on an enlarged θ-interval, namely $\frac{\omega_0}{\omega_r} \cdot \left[\frac{\pi}{2} - \frac{2\rho}{r_0}, \frac{3\pi}{2} + \frac{2\rho}{r_0}\right]$. This shape wave deformation continues in the same direction by increasing the energy of the particle, in such a way that the motion verifies $r(\theta) < r_0$ for the biggest part of the θ-interval $\frac{\omega_0}{\omega_r} \cdot [0, 2\pi]$, except for a very narrow region where $r(\theta)$ completes the oscillation and reaches $r_{\max}$ with a sharp peak in the (θ, r) representation.

2.4   Large oscillations

For $h \gg h_0$, large oscillations in the radial coordinate are obtained (see Fig. 3). The trajectories become ovoid-like reaching the maximum radial velocity $\dot{r}_{max}$ for $r = r_0$: $\dot{r}_{max} = \dot{r}(r_0)$. As $r_{max} \gg r_{min}$, the relation (18) simplifies to

$$2\, r_{min}^2 r_{max} \simeq r_0^3.$$

Finding the relation between the radial and angular frequencies requires a more elaborated calculation in this case. We proceed to integrate Eqs. (14) and (13) when $h \to \infty$, in order to obtain the asymptotic value of ω_r/ω_θ.

We consider a half-oscillation between the extreme points, r_{min} and r_{max}, of an orbit. We divide this interval into two parts: (r_{min}, r_0) and (r_0, r_{max}), which are covered in the time intervals: $(0, t_1)$ and (t_1, t_2), respectively. As the variables in the radial equation (14) are separated, we write

$$\int_0^{t_2} dt = \int_{r_{min}}^{r_{max}} \frac{dr}{\sqrt{2(h - V(r))}} = \int_{r_{min}}^{r_{max}} \frac{r\, dr}{\sqrt{2hr^2 - 2Br^3 - A}}. \tag{23}$$

This elliptic integral cannot be put in elemental functions. When $h \to \infty$, the

potential $V(r)$ behaves as

$$V(r) \simeq V_1(r) = \frac{A}{2r^2} \quad \text{for} \quad r_{min} < r < r_0,$$

$$V(r) \simeq V_2(r) = Br \quad \text{for} \quad r_0 < r < r_{max},$$

and, r_{min} and r_{max} are now easily calculated,

$$V_1(r_{min}) \simeq h \;\Rightarrow\; r_{min} \sim \sqrt{\frac{A}{2h}},$$

$$V_2(r_{max}) \simeq h \;\Rightarrow\; r_{max} \sim \frac{h}{B}.$$

Under these assumptions an approximated value of integral (23) is obtained. The error is basically concentrated in the region $(r_0 - \Delta r, r_0 + \Delta r)$. However, because velocity is maximal in r_0 and tends to infinity when $h \to \infty$, the time invested by the system in that region tends to zero and thus the error vanishes.

Substituting $V(r)$ by $V_1(r)$ and integrating expression (23) in the interval (r_{min}, r_0), we get

$$t_1 \simeq \frac{1}{\sqrt{2h}} \sqrt{r_0^2 - r_{min}^2} \;\overset{h \to \infty}{\longrightarrow}\; \frac{r_0}{\sqrt{2h}}. \tag{24}$$

Making $V(r) \simeq V_2(r)$ in the region (r_0, r_{max}), the integral (23) is approximated by

$$t_2 - t_1 \simeq \sqrt{\frac{2}{B}} \sqrt{r_{max} - r_0} \;\overset{h \to \infty}{\longrightarrow}\; \frac{\sqrt{2h}}{B}. \tag{25}$$

Thus the system spends most of the oscillation time in the upper region of the circular orbit $r = r_0$, and it tends to infinity when $h \to \infty$.

The angular part of the motion is derived from Equation (13),

$$\int_0^{\theta_2} d\theta = \frac{L_z}{m \sin^2 \phi_0} \int_0^{t_2} \frac{dt}{r^2(t)}, \tag{26}$$

where the integral is performed in the angular regions $(0, \theta_1)$ and (θ_1, θ_2) which are covered when time runs in the intervals $(0, t_1)$ and (t_1, t_2), respectively.
Taking into account the relation

$$r^2(t) \simeq 2ht^2 + r_{min}^2 \quad \text{for} \quad 0 < t < t_1,$$

and, using the value of $t = t_1$ given by expression (24), we obtain θ_1 when $h \to \infty$:

$$\theta_1 = \frac{\pi}{2 \sin \phi_0}. \tag{27}$$

The behavior in the second interval (θ_1, θ_2) is quite different. Starting from the relation

$$t - t_1 \simeq \sqrt{\frac{2}{B}} \left(\sqrt{r_{max} - r_0} - \sqrt{r_{max} - r} \right) \quad \text{for} \quad t_1 < t < t_2,$$

$r(t)$ can be derived. As $\frac{Bt}{\sqrt{2h}} < 1$, expression (26) is approximately integrated in the interval (t_1, t_2). We get in the limit $h \to \infty$:

$$\theta_2 - \theta_1 \simeq \frac{1}{r_0} \sqrt{\frac{2}{h}}, \tag{28}$$

which shows that for large energies the dynamics for $r > r_0$ is projected onto a tight peak in the (θ, r) plane. Thus most of the part of the θ-coordinate is covered when the particle is under the circular orbit $r = r_0$ although the system spends its time essentially over that circular orbit (see Fig. 5).

Summarizing, if $\frac{T_r}{2} \simeq t_2$ is the semi-period of the radial oscillation, the frequencies w_r and ω_θ for large energies are

$$\omega_r = \frac{\pi}{T_r/2} \quad \simeq \quad \frac{2\pi}{t_2},$$

$$\omega_\theta = \frac{\theta_2}{T_r/2} \quad \simeq \quad \frac{\pi/\sin \phi_0}{t_2},$$

which implies that, when $h \to \infty$, the frequency ratio is

$$\frac{\omega_r}{\omega_\theta} = 2 \sin \phi_0. \tag{29}$$

Comparing this last expression with the frequency ratio (21) for small radial oscillations, a general behavior of this quantity can be advanced:

$$\frac{\omega_r}{\omega_\theta} = k \sin \phi_0 \quad \text{with } k \text{ in the range} \quad \left\{ \begin{array}{c} \sqrt{3} < k < 2 \\ \updownarrow \quad \updownarrow \quad \updownarrow \\ h_0 < h < \infty \end{array} \right. . \tag{30}$$

This relation will be computationally and analytically discussed in the next section.

3   Universal Relations of the Dynamics

3.1   Universal equations

As has been established in the last section, the evolution of the particle sliding inside the cone is governed by Eq. (14) in its radial part and by Eq. (13) in its angular one. These equations are expressed in a form which depends on the gravitational field and surface geometrical properties and, evidently, on the initial value of the global dynamical constants. We proceed now to write them in a universal form by the rescaling of radial, angular and time variables. In short, all information on the dynamics will be contained in these reduced equations which are independent of any 'external' information. To obtain the real trajectory from these equations it will be enough to undo the change of variables with the 'external' information: mass, geometry, field and initial conditions.

If we substitute the value of the vertical angular momentum, L_z^2, given by Eq. (3) and we perform the change of variable: $\tilde{r} = r/r_0$, radial and angular equations are reduced to

$$\ddot{\tilde{r}} + \frac{\omega_{r,h_0}^2}{3}(1 - \tilde{r}^{-3}) = 0, \tag{31}$$

$$\dot{\theta}^2 - \omega_0^2 \tilde{r}^{-4} = 0, \tag{32}$$

where ω_{r,h_0}^2 and ω_0^2 take the values given by Eqs. (5) and (4), respectively. Rescaling the time variable by $\tilde{t} = \frac{\omega_{r,h_0}}{\sqrt{3}}t$ and the angular variable by $\tilde{\theta} = \theta \sin \phi_0$, the *non-dimensional* equations of the motion are found:

$$\ddot{\tilde{r}} + (1 - \tilde{r}^{-3}) = 0, \tag{33}$$

$$\dot{\tilde{\theta}} - \tilde{r}^{-2} = 0. \tag{34}$$

Remarkably, in this form, the equations apparently lose any characteristic of the system. Every possible trajectory on the conical surface is projected into a solution

of these equations and, in that sense, we say that they are universal. They contain all the information about the dynamical behavior of the particle.

In particular, $\tilde{r} = 1$ is the singularity representing the circular orbit and any other orbit runs between the extreme values, $\tilde{r}_{min}$ and $\tilde{r}_{max}$, which satisfy: $0 < \tilde{r}_{min} < 1$ and $1 < \tilde{r}_{max} < \infty$. Expression (18) is now rewritten as

$$\frac{2\,\tilde{r}_{min}^2 \tilde{r}_{max}^2}{\tilde{r}_{min} + \tilde{r}_{max}} = 1. \tag{35}$$

3.2 Universal frequency relation

From the universal equations of the dynamics, we remake now all the calculations presented in Subsection 2.4 to show the universal relation (30). For simplicity in the notation, we rename the variables: $\tilde{r} \to r$ and $\tilde{t} \to t$.

The energy conservation becomes in the new coordinates

$$\frac{\dot{r}^2}{2} + \tilde{V}(r) = E \tag{36}$$

where $\tilde{V}(r) = r + \frac{1}{2r^2}$ and $E = \frac{h}{gr_0 \cos \phi_0}$. Hence, in this representation, the dynamics settles in the circular orbit when $E = 3/2$, and, small or large oscillations are obtained when the value of the normalized energy E runs on the interval $(\frac{3}{2}, \infty)$. For an arbitrary energy $E \in (\frac{3}{2}, \infty)$, r_{min} is deduced from the equality:

$$\tilde{V}(r_{min}) = E \quad \to \quad r_{min} = r_{min}(E).$$

The radial coordinate as function of the time, $r(t)$, is calculated after performing the integral:

$$\int_0^t dt = \int_{r_{min}}^r \frac{rdr}{\sqrt{2Er^2 - 2r^3 - 1}} \quad \to \quad r = r(t, E).$$

For $t = \frac{T_r}{2}$, with T_r the semi-period of the radial oscillation, r_{max} is reached. Recall that $r_{max}(E)$ is obtained from Expression (35). Thus, we get $T_r(E)$ by solving the

equation

$$r_{max}(E) = r\left(\frac{T_r}{2}, E\right) \quad \rightarrow \quad T_r = T_r(E).$$

The angle $\tilde{\theta}_2$ covered by the particle during a radial semi-period is given by

$$\tilde{\theta}_2 = \int_0^{\tilde{\theta}_2} d\tilde{\theta} = \int_0^{T_r/2} \frac{dt}{r^2(t)} \quad \rightarrow \quad \tilde{\theta}_2 = \tilde{\theta}_2(E).$$

Therefore the frequency ratio obeys the following relation:

$$\frac{\omega_r}{\omega_\theta} = \frac{\pi/\frac{T_r}{2}}{\tilde{\theta}_2/\frac{T_r}{2}} = \frac{\pi}{\tilde{\theta}_2} \quad \sin\phi_0 = k(E)\sin\phi_0, \tag{37}$$

where

$$k(E) = \left[\frac{1}{\pi}\int_0^{\frac{T_r(E)}{2}} \frac{dt}{r^2(t, E)}\right]^{-1}. \tag{38}$$

As we calculated in the previous section,

$$k\left(E = \frac{3}{2}\right) = \sqrt{3},$$
$$k(E = \infty) = 2,$$

and as can be deduced from numerical calculations of $k(E)$ (see Fig. 6), we claim that

$$\sqrt{3} < k(E) < 2 \quad \text{when} \quad \frac{3}{2} < E < \infty, \tag{39}$$

which allows us to approximate integral (38), valid within 13%, by

$$\int_0^{\frac{T_r}{2}} \frac{dt}{r^2(t)} \simeq \frac{\pi}{2},$$

for any trajectory $r(t)$ of arbitrary energy E.

3.3   Universal dynamical relation

We proceed to convert the almost universal frequency ratio (30) in a dynamical relation. If N represents the number of revolutions that the particle performs around

the cone in the half of a radial oscillation, then $\theta_2 = 2\pi N$, and expression (37) is now written as follows:

$$\frac{\omega_r}{\omega_\theta} = \frac{\pi/\frac{T_r}{2}}{2\pi N/\frac{T_r}{2}} = \frac{1}{2N} = k \sin \phi_0. \tag{40}$$

Taking the good approximation $k \simeq 2$, we obtain the universal dynamical relation

$$4N \sin \phi_0 \quad \simeq \quad 1. \tag{41}$$

In the case of small cone opening, $\phi_0 \ll 1$, this simplifies to

$$4N\phi_0 \quad \simeq \quad 1, \tag{42}$$

which establishes a simple experimental test to find out if a particle describing a trajectory within a conical-like surface is truly governed by the equations under study. That is, if the particle's dynamics is equivalent to sliding without friction on the surface.

3.4   Universal geometrical relation

Following the same line of reasoning as in Section 3.3 and motivated by the functioning of the industrial 'cyclone' gas-particle separators [4, 5], suppose a motion of the system developing on a truncated cone such as that drawn in Fig. 7. The geometrical lengths of the orbit are the maximum and minimum radii, R_{min} and R_{max}, and the vertical height, H, given by

$$R_{max} = r_{max} \sin \phi_0,$$

$$R_{min} = r_{min} \sin \phi_0$$

$$H = \frac{R_{max} - R_{min}}{\tan \phi_0}.$$

By replacing the values of r_0 and L_z,

$$L_z = mR_{min}v_{down},$$

$$r_0^3 = \frac{1}{g \cos \phi_0} \left(\frac{R_{min}v_{down}}{\sin \phi_0} \right)^2,$$

in the relation (18), we find a universal geometrical requirement for an arbitrary trajectory

$$\frac{R_{max}^2}{R_{min} + R_{max}} = \frac{v_{down}^2 \tan \phi_0}{2g}, \tag{43}$$

where v_{down} is the linear velocity of the particle at the bottom of its orbit. Conservation of L_z implies that

$$\frac{R_{max}}{R_{min}} = \frac{v_{down}}{v_{up}}, \tag{44}$$

with v_{up} the linear velocity at the top of the motion. Expression (43) can also been written as follows:

$$\frac{R_{min}^2}{R_{min} + R_{max}} = \frac{v_{up}^2 \tan \phi_0}{2g}. \tag{45}$$

These universal geometrical relations can be also used to decide if a physical phenomenon has an underlying dynamics such as that studied here.

3.5 Energy transformation

The kinetic energy $\mathcal{T}_\theta$ of the angular part of the motion can be written as

$$\mathcal{T}_\theta = \frac{1}{2} L_z \dot{\theta},$$

and the conservation of energy at a point $P(r, \theta)$ of a trajectory is the result of the balance between kinetic and potential energy,

$$-\frac{1}{2} m \dot{r}^2 + \frac{1}{2} L_z (\dot{\theta}_{down} - \dot{\theta}) = mg\Delta z, \tag{46}$$

where $\dot{r}$ and $\dot{\theta}$ are the radial and angular velocities at P, $\quad \Delta z = (r - r_{min}) \cos \phi_0$ is the vertical distance from P to the bottom of the trajectory and $\dot{\theta}_{down}$ is the angular velocity at the bottom of the motion.

If we rename $v_r = \dot{r}$ and $v_{down} = R_{min}\dot{\theta}_{down}$, Δz is obtained from the latter expression,

$$\Delta z = -\frac{v_r^2}{2g} + \frac{v_{down}^2}{2g} \left(1 - \left(\frac{R_{min}}{R}\right)^2\right), \tag{47}$$

where $R = r \sin \phi_0$.

The vertical distance, $H = \Delta z(R_{max})$, covered by the particle in its motion verifies

$$H = \frac{v_{down}^2}{2g} \left(1 - \left(\frac{R_{min}}{R_{max}} \right)^2 \right).$$

(48)

Observe that not all the kinetic energy can be transferred to potential energy at the top of the orbit. A percentage of the total energy, given by the factor $\gamma = \left(\frac{R_{min}}{R_{max}} \right)^2$, remains as angular kinetic energy in order to conserve L_z. This transference is more efficient when the geometry of the system approaches that of a plane system because L_z can be conserved by the large radial component of the trajectory. This happens when ϕ_0 tends to $\frac{\pi}{2}$. On the contrary, the transfer of energy is nearly forbidden and is totally inefficient when the system tends to the cylindrical form, that is, when ϕ_0 goes to zero.

The interesting energy-transfer process acting in this system, between the angular kinetic term, the radial kinetic term and the potential energy term is not habitual in simple mechanical systems of only two degrees of freedom. This type of idea typically appears later when dealing with considerably more complex systems of three or more degrees of freedom, such as the symmetric top.

Note also that in other problems of a point-like particle sliding under gravity on surfaces such as planes or cylinders, this rich energy-transfer does not occur. There the transfer is as in a bullet motion, i.e., between the 'vertical' kinetic term and the potential energy term. In the case of sliding on the inside of a spherical surface the energy transfer process occurs as in the cone but with two important differences. First, in the sphere, no matter the value of the energy, the spatial range of the orbits is limited to the size of the system. And, second, when the particle crosses the equatorial line, the vertical reaction of the surface points downward and

the particle may lose contact with the surface.

For all these reasons, we believe that the cone problem has a great academic potential and deserves adequate attention in General Physics courses.

4 Conclusions

The understanding of the physical mechanisms of many complicated phenomena requires a drastic simplification in a first approach. Here we have isolated two basic ingredients, the conical geometry and the gravitational field, present in different natural formations such as volcanoes or tornadoes, and in a series of technical devices such as pipes, particle precipitators or centrifugal separators.

We have studied the evolution of a particle sliding on the inside of a conical surface, with opening angle ϕ_0, under the action of gravity. This conservative system has two constants of motion, the total energy, $\mathcal{H}$, and the vertical component of the angular momentum, L_z, which makes it integrable. The initial values of both constants determine the trajectory described in its four-dimensional phase space, which in general is quasiperiodic and it fills up a 2-torus.

Every orbit is closed when it is projected in the radial plane $(r, \dot{r})$, running between the two extreme values, r_{min} and r_{max}. For a quasiperiodic trajectory, the oscillations in the angular part are not synchronized with the radial ones. It means that wherever the particle is released on the inside of the cone, it will come back as closely as desired to the same point. Only when the relation between the frequencies of these two (radial and angular) uncoupled modes is a rational number, p/q, we will have a periodic motion. This is satisfied when $\frac{p}{q} = k(E) \sin \phi_0$, where $k(E)$ is a parameter dependent only on the product $E \sim \mathcal{H} L_z^{-2/3}$ (see eqs. (30) and (36)) and

given by the expression (38). The range of $k(E)$ has been numerically calculated in Fig. 6 and it varies in the interval $\sqrt{3} < k(E) < 2$, where the value $k = \sqrt{3}$ is reached for small oscillations about the circular motion and $k = 2$ when the energy of the particle tends to infinity.

Several universal relations characterizing the dynamics of the system have been obtained. First, the non dimensional equations have been derived by applying different length and time rescaling transformations. Independently of the initial conditions, all the information about the motion is contained in these *universal equations*. Second, the relation between the radial and angular frequencies has been shown to depend only on the parameter $k(E)$. It has been reinterpreted as an *universal dynamical relation* of the number of revolutions that the particle performs around the cone for each radial oscillation. Third, every trajectory develops on a truncated part of the cone. The characteristic lengths and extreme velocities defining the orbit satisfy a *universal geometrical relation*. Finally, some relations of the transfer of energy between the orbital, radial and potential energy have been presented.

The application of these relations to some realistic systems can be used to test if the corresponding underlying dynamics is understandable as caused by the combined action of the gravitational field and a force perpendicular to the motion, which could be identified as the reaction force of the conical profile associated with the phenomenon under study. It is our aim to explore this direction in a future work.

A Appendix: Periodic Trajectories of the Motion

Periodic motions of a dynamical system are represented by closed orbits. They play a fundamental role in understanding the topology and orbit organization of the phase space, but there are no general methods to find them. Here we sketch a procedure that simplifies the localization of periodic orbits in the present system. It consists of an adequate projection on a phase space of a lower dimension, where the property of periodicity is conserved.

By a *conserving periodicity projection* we mean that if an orbit is periodic in the space of projection then it is also periodic in the original higher dimensional system. If so, there exists a bi-univocal relation between the periodic orbits of both spaces, and it is enough to find them in the space of projection. Thus the problem is enormously simplified.

The motion of the particle inside the cone is governed by four autonomous differential equations of first order, given by Hamilton's equations of the system:

$$\dot{r} = \frac{\partial \mathcal{H}}{\partial p_r} = \frac{p_r}{m}, \tag{49}$$

$$\dot{\theta} = \frac{\partial \mathcal{H}}{\partial p_\theta} = \frac{p_\phi}{mr^2 \sin^2 \phi_0}, \tag{50}$$

$$\dot{p}_r = -\frac{\partial \mathcal{H}}{\partial r} = \frac{-p_\phi^2}{mr^3 \sin^2 \phi_0} - mg \cos \phi_0, \tag{51}$$

$$\dot{p}_\theta = -\frac{\partial \mathcal{H}}{\partial \theta} = 0. \tag{52}$$

Hence a trajectory of the system evolves in the four-dimensional phase space with coordinates $(r, \theta, p_r, p_\theta)$, or, equivalently, with coordinates $(r, \theta, \dot{r}, \dot{\theta})$ if the change of variables $(p_r, p_\theta) \rightarrow (\dot{r}, \dot{\theta})$, given by Eqs. (10-11), is performed.

As the dynamical equations fulfil the regular conditions of continuity and smoothness, the field is uniquely determined on each point, and the flow has no singularities in the region of phase space with physical significance. That is, those points whose

coordinates verify: $r > 0$, $0 \leq \theta < 2\pi$ and $0 < \phi_0 < \frac{\pi}{2}$.

A phase path will be closed, and then periodic, when it visits the same point twice and in consequence, an infinite number of different times:

$$\text{A trajectory is closed} \iff \exists T > 0 \rightarrow (r, \theta, \dot{r}, \dot{\theta})_{t=0} = (r, \theta, \dot{r}, \dot{\theta})_{t=T}.$$

If we perform the projection $(r, \theta, \dot{r}, \dot{\theta}) \rightarrow (r, \dot{r})$, the integral curves on the new two-dimensional phase space are solutions of Eq. (15) and they are closed orbits as can be seen, for instance, in Fig. 3. Therefore, for this particular projection, all the four-dimensional trajectories are projected on two-dimensional periodic orbits. This is a non conserving periodicity projection.

By simple inspection of the problem it is obvious that periodic orbits of the equations system (49-52) are those verifying $(r(t), \theta(t)) = (r(t+T), \theta(t+T))$. Take the origin of time, $t = 0$, when the trajectory passes through one of the extreme points, r_{min} or r_{max}, of the radial coordinate. At these turning points, the radial velocity vanishes, $\dot{r} = 0$. As the angular velocity around the cone $\dot{\theta}$ depends only on the dynamical constant L_z of the motion and on the radial coordinate r (see Eq. (13)), it follows that $(r, \theta, \dot{r} = 0, \dot{\theta})_{t=0} = (r, \theta, \dot{r} = 0, \dot{\theta})_{t=T}$, and the trajectory is also closed in the total four-dimensional space.

Therefore, in order to find the periodic motions of the whole four-dimensional system we must study the conserving periodicity projection: $(r, \theta, \dot{r}, \dot{\theta}) \rightarrow (r, \theta)$. The periodic orbits are in this case all those verifying that the ratio between its radial, ω_r, and, angular, ω_θ, frequencies is a rational number:

$$\text{Periodic motion} \quad \Leftrightarrow \quad \frac{\omega_r}{\omega_\theta} = \frac{p}{q} \quad \text{with } p, q \in \mathbb{Z}. \tag{53}$$

Note that this conclusion is well-established in literature [12], and is a consequence of the integrability of the Hamiltonian. The present four-dimensional system has two independent global constants, namely, the energy, $\mathcal{H}$, and the vertical component of

the angular momentum, L_z, which allow us to rewrite the Hamiltonian in action-angle coordinates. Hence the trajectories are 2-frequency quasi-periodic and, in general, they fill up a 2-torus. The angular velocities specifying the motion on the 2-torus are ω_r and ω_θ, and the condition for having a periodic orbit is given by Eq. (53).

References

[1] D.L. Turcotte and G. Schubert, *Geodynamics: Applications of Continuum Physics to Geological Problems*, Section 9.6, John Wiley & Sons (1982).

[2] *The Oxford Companion to the Earth*, Edited by P.L. Hancock and B.J. Skinner, page 1089, Oxford University Press (2000).

[3] H. Bluestein, *Tornado Alley: Monster Storms of the Great Plains*, Oxford University Press (1999).

[4] C. B. Lalor and A. J. Hickey, "Pharmaceutical aerosols for delivery of drugs to the lungs", in *Physical and Chemical Properties of Aerosols*, pag.391, Edited by I. Colbec, Blackie Academic & Professional (1998).

[5] G. M. Masters, *Introduction to Environmental Engineering and Science*, 2nd edit. Prentice Hall Inc. (1998).

[6] R. Xiang, S.H. Park and K.W. Lee, "Effects of cone dimension on cyclone performance", Aerosol Science **32**, 549-561 (2001).

[7] O. Al-Hawaj, "A numerical study of the hydrodynamics of a falling liquid film on the internal surface of a downward tapered cone", Chemical Engineering Journal **75**, 177-182 (1999).

[8] G.K. Batchelor, "An Introduction to Fluid Dynamics", section 7.8, Cambridge University Press (1994).

[9] P. Bak, C. Tang, and K. Wiesenfeld, "Self-organized criticality: an explanation of the 1/f noise", Phys. Rev. Lett. **59**, 381384 (1987).

[10] T.W.B. Kibble, *Classical Mechanics*, page 263, Longman Scientific & Technical (1985).

[11] A.A. Andronov, A.A. Vitt and S.E. Khaikhin, *Theory of Oscillators*, Dover Publications (1996).

[12] H. Goldstein, *Classical Mechanics*, Addison-Wesley, Reading (1980).

Figure Captions

1. Geometrical coordinates (r, θ) defining the dynamics of the particle of mass m, which slides under the action of the gravitational field g on the inside of a cone with opening angle ϕ_0.

2. Plot of the effective radial potential energy, $U(r)$, for $L_z = 1$, $\phi_0 = \frac{\pi}{4}$ and m=1.

3. Pattern of solutions of the radial equation (14) for $\phi_0 = \frac{\pi}{4}$. These are closed curves between r_{min} and r_{max} taking the maximal radial velocity for $r = r_0$. **(a)** $L_z = 0.1$, $r_0 = 0.142$; **(b)** $L_z = 1$, $r_0 = 0.660$; **(c)** $L_z = 10$, $r_0 = 3.066$.

4. Small radial oscillations about the circular motion for $L_z = 1$: $r \sim r_0 + \Delta r$, $\Delta r_{max} \sim 0.05$. **(a)** $\omega_r > \omega_0$ for $\phi_0 = \frac{\pi}{4} > \bar{\phi}_0$, $r_0 = 0.660$. **(b)** $\omega_r < \omega_0$ for $\phi_0 = \frac{\pi}{6} < \bar{\phi}_0$, $r_0 = 0.778$.

5. Large radial oscillations about the circular motion for $L_z = 1$: $r \sim r_0 + \Delta r$, $\Delta r_{max} \sim 1.5$. **(a)** $\omega_r > \omega_\theta$ for $\phi_0 = \frac{\pi}{4} > \bar{\phi}_0$, $r_0 = 0.660$. **(b)** $\omega_r < \omega_\theta$ for $\phi_0 = \frac{\pi}{6} < \bar{\phi}_0$, $r_0 = 0.778$.

6. Numerical calculation of $k = \frac{\omega_r}{\omega_\theta \sin \phi_0}$ as a function of $r\_relation$: **(+)** $r\_relation = \frac{r_{min}}{r_0}$ and **(x)** $r\_relation = \frac{r_0}{r_{max}}$. This curve is independent of L_z and ϕ_0. In this case, the computation has been performed for $L_z = 1$ and $\phi_0 = \frac{\pi}{4}$.

7. Characteristic lengths of a typical orbit developing on the truncated cone defined by R_{min}, R_{max} and H.